Chapter 5

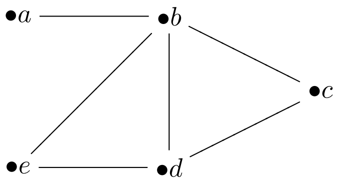
Topics in Graph Theory

Graphs are a quite common mathematical term which we may encounter in simple geometry or Calculus, and particularly we have seen them as the digraphs of binary relations in Section 3.1.2. Modern graph theory, however, has developed into a very advanced study which has many powerful applications. This chapter is but a shallow survey into some of the most familiar topics in the subject.

5.1 Some Basic Features

Definition. A *graph* G is a composite of a finite set V_G of *vertices* and another set E_G of *edges*, where an edge is defined to be a set of two distinct vertices. When there is no ambiguity, we write V and E, instead of V_G and E_G, respectively. If $E = \emptyset$, we have a *trivial graph* of only vertices.

A graph can be represented by a picture in very much the same way we draw vertices and edges of a digraph. For example, the picture given below



(5.1)

represents the graph G with the set of vertices $V = \{a, b, c, d, e\}$ and edges $E = \{\{a, b\}, \{b, c\}, \{b, d\}, \{b, e\}, \{c, d\}, \{d, e\}\}$. Unlike digraphs, however, note that here an edge has no direction—thus we use set notation $\{a, b\}$ in place of the ordered pair (a, b) to represent an edge.

5.1.1 Degrees

Definition. For brevity, we denote the edge $\{a, b\} \in E$ simply by ab, if doing so does not cause misunderstanding. In particular when $ab \in E$, we say that the vertices a and b are *adjacent*. We define the *degree* of a vertex a, written $\deg(a)$, to be the number of vertices in G which are adjacent to a. Moreover, let $\deg G = \sum_{a \in V_G} \deg(a)$, the *degree* of the graph G.

In our example of G given by (5.1) earlier, we have $\deg(a) = 1$, $\deg(b) = 4$, $\deg(c) = 2$, $\deg(d) = 3$, $\deg(e) = 2$, and $\deg G = 1 + 4 + 2 + 3 + 2 = 12$.

Theorem 5.1 (Euler's Theorem)**.** The degree of any graph G is twice the number of its edges, i.e., $\deg G = 2\,|E_G|$. In particular, the degree of any graph is an even number.

Proof. Every edge $ab \in E$ contributes two toward the degree sum, one for $\deg(a)$ and another for $\deg(b)$, thus the result. $\bigtriangledown$

If we replace E_G by a multiset, allowing repetition of edges, then G will be called a *multigraph*. Sometimes a vertex in a multigraph is allowed to be adjacent to itself, creating a *loop*, i.e., an element $aa \in E$. Multigraphs, however, are not a concern in this text.

Exercise* 5.1. Show that Euler's theorem holds for multigraphs as well, if $\deg(a)$ is defined to be the number of edges containing a, counting multiplicity and duplicity of each loop.

Definition. The following four families of particularly interesting and useful graphs will each be given a special name.

1) A *complete* graph K_n is a graph with n vertices, all of which are adjacent one to another. In particular, K_3 is also called a *triangle*.

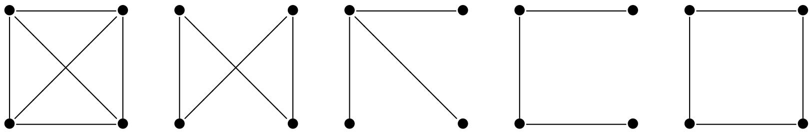
2) The $m+n$ vertices of the graph $K_{m,n}$ are bipartitioned—i.e., partitioned into two subsets—into m and n elements each, such that two vertices are adjacent if and only if they do not belong together. A graph with this property is called *complete bipartite*.

3) A *path* P_n is a graph with n vertices, $V = \{v_1, v_2, \ldots, v_n\}$, and $n-1$ edges, $E = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}$. With this notation, we say that P_n is a path from v_1 to v_n.

4) For $n \geq 3$, let $C_n = P_n \cup \{v_nv_1\}$, which we call a *closed path* or a *cycle* of n vertices.

Question. Can you name five mathematical terms given throughout this text, one in each chapter, which begin with the prefix *bi-*?

We illustrate below possible drawings of some of these special graphs.



K_4 $K_{2,2}$ $K_{1,3}$ P_4 C_4

Question. Can you spot two graphs among the five pictured above, which are essentially identical sets of vertices and edges?

Exercise 5.2. Evaluate the degree of each graph K_n, $K_{m,n}$, P_n, and C_n, and count the number of edges in each graph as well.

Exercise* 5.3. Prove that the number of edges in $K_2, K_3, K_4, \ldots$ form the sequence of the *triangular numbers,* i.e., $1, 1+2, 1+2+3, \ldots$

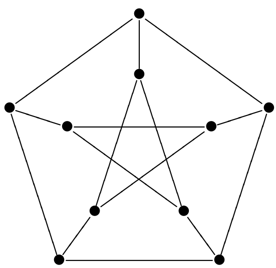
Test 5.4. Which one of these four graphs has the most number of edges?

a) K_{99}
b) $K_{50,50}$
c) P_{100}
d) C_{200}

Test 5.5. The degree of a complete bipartite graph is 210. What could be the smallest number of vertices this graph has?

a) 15
b) 22
c) 26
d) 38

Definition. The *Peterson graph* consists of two copies of C_5 which are interconnected to each other in the way depicted below.



Question. What is the degree of the Peterson graph?

Definition. A graph is called *regular* if all its vertices have equal degrees, otherwise *irregular.* In particular, a regular graph in which $\deg(a) = d$ for every vertex a, is called *d-regular.* For example, K_3 is 2-regular, whereas P_5 is irregular.

Question. Is the Peterson graph regular?

Test 5.6. Which one of these four graphs is not regular?

a) K_{99}
b) $K_{99,99}$
c) P_{99}
d) C_{99}

Exercise 5.7. Analyze the regularity of the graphs K_n, $K_{m,n}$, P_n, and C_n.

Definition. If $V_G = \{v_1, v_2, \ldots, v_n\}$, the *degree sequence* of the graph G is the sequence $\{\deg(v_i)\}$ of length n, arranged in decreasing order. For example, the degree sequence of P_5 is $2, 2, 2, 1, 1$.

Question. What is the degree sequence of the Peterson graph?

Exercise 5.8. Find the degree sequence of each K_n, $K_{m,n}$, P_n, and C_n.

We call a degree sequence such as $2, 2, 2, 1, 1$ *graphical* since there exists a graph, e.g., P_5, having this degree sequence. Obviously, not all degree sequences are graphical; for instance, $4, 3, 2, 2, 2$ cannot possibly be graphical, says Euler's theorem. (Why?) The following algorithm can be used to determine whether a given degree sequence is graphical.

Question. Why is $4, 3, 2, 1$ not graphical either?

Algorithm 5.2 (Graphical Degree Sequence)**.** Given a degree sequence, we determine graphical or not graphical.

1) Delete the first integer, say k.

2) From what remains, subtract the first k numbers each by 1. If this is not possible, the sequence is not graphical. If we get all zeros, the sequence is graphical.

3) If necessary, rearrange the newly formed sequence in decreasing order, and repeat the above steps until a conclusion is obtained.

Observation. In each round, if the resulting new sequence is graphical, so is the preceding one, by simply restoring the deleted vertex and its associated edges. To prove the converse is not that trivial, though. ▽

Example. Determine if $3, 2, 2, 1, 1, 1$ is a graphical degree sequence.

Solution. Remove the first number, 3, and subtract the first three numbers of what remains each by 1. We get $1, 1, 0, 1, 1$. The complete iterations recorded below show that the sequence is graphical.

$$3, 2, 2, 1, 1, 1 \to 1, 1, 0, 1, 1 \to 1, 1, 1, 1, 0 \to 0, 1, 1, 0 \to 1, 1, 0, 0 \to 0, 0, 0$$

Exercise 5.9. Apply the algorithm to these sequences. If a sequence is graphical, draw a graph satisfying the given degree sequence.
a) $3, 2, 2, 1, 1, 1$
b) $4, 3, 3, 2, 1, 0$
c) $5, 3, 2, 2, 1, 1$
d) $5, 4, 4, 3, 3, 3, 3, 2, 2, 1$

Exercise* 5.10. For multigraphs, show that a degree sequence is graphical if and only they sum to an even number.

5.1.2 Isomorphisms

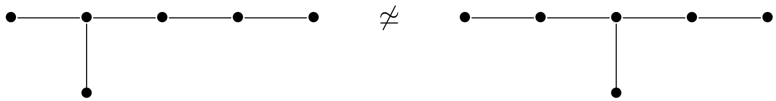
Definition. Two graphs are *isomorphic* to each other, written $G \simeq H$, if there is a bijection $f : V_G \to V_H$ such that a and b are adjacent in G if and only if $f(a)$ and $f(b)$ are adjacent in H.

For example, $K_3 \simeq C_3$, so both are called triangles. Note also that $K_2 \simeq K_{1,1} \simeq P_2$. Being isomorphic means that the two graphs can be represented by identical pictures, if we discount the labeling of the vertices.

Test 5.11. Which one of these graph isomorphisms is false?
a) $K_{1,2} \simeq P_3$
b) $K_{1,3} \simeq P_4$
c) $K_{2,2} \simeq C_4$
d) All the above are true statements.

If $G \simeq H$, then clearly G and H must have the same number of vertices, the same number of edges, and identical degree sequences. However, it is important to note that none of these properties is a sufficient condition for isomorphism.

Example. These two graphs given below have each 6 vertices, 5 edges, and degree sequence $3, 2, 2, 1, 1, 1$. Prove that they are not isomorphic to each other.



Solution. Any bijection between them must preserve the unique vertex of degree 3. Observe that on the left, that unique vertex is adjacent to three others with degrees $2, 1, 1$. On the right, however, the adjacent three have degrees $2, 2, 1$. This proves that no such bijection will work.

Question. Is there another graph, isomorphic to neither of the above two, but with the same degree sequence $3, 2, 2, 1, 1, 1$?

Exercise 5.12. Find several (two to four) non-isomorphic graphs with each given degree sequence.

a) $4, 4, 3, 2, 2, 1$
b) $2, 2, 2, 2, 2, 1, 1$
c) $5, 3, 2, 2, 1, 1, 1, 1$
d) $3, 3, 3, 3, 3, 3, 3, 3$

Definition. A graph H is a *subgraph* of the graph G if $V_H \subseteq V_G$ and $E_H \subseteq E_G$. We denote this relation by $H \subseteq G$, and say that G *contains* H.

By abuse of notation, we may write $H \subseteq G$ when we really mean that G contains a subgraph which is isomorphic to H. Hence, for example, $P_3 \subseteq C_3$ and $K_{2,2} \subseteq K_{2,4}$.

Test 5.13. Which one of these four graphs does not contain C_4?

a) K_5
b) $K_{5,5}$
c) $K_{3,5}$
d) C_5

Question. How many non-isomorphic cycles are contained in the Peterson graph?

Definition. A graph G is called *connected* if there is a path from any vertex to any other vertex in G, otherwise *disconnected*. A *component* of G is then a maximal connected subgraph of G.

Hence, a graph is disconnected if and only it has more than one component. It is clear that isomorphic graphs must have the same number of components.

Exercise 5.14. If G is connected, prove that $|E_G| \geq |V_G| - 1$.

Definition. An edge in a connected graph G is a *bridge* when G would become disconnected if this edge be removed.

Question. Does the Peterson graph have a bridge?

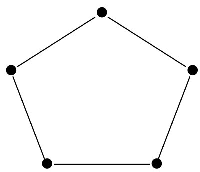
Test 5.15. Which one of these four graphs contains a bridge?

a) K_9
b) $K_{2,9}$
c) P_9
d) C_9

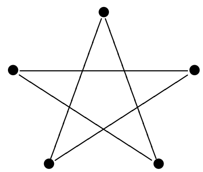
Exercise 5.16. Determine exactly when each of the graphs K_n, $K_{m,n}$, P_n, and C_n, may contain a bridge.

Definition. The *complement* of a graph G is the graph $\overline{G}$, where $V_{\overline{G}} = V_G$ and $ab \in E_{\overline{G}}$ if and only if $ab \notin E_G$.

Below we show the picture of C_5 next to its complement.



$C_5 \quad \to \quad \overline{C}_5$



Exercise 5.17. Draw the complements of P_4, C_4, K_5, and of $K_{5,4}$.

Question. What is the degree of the complement of the Peterson graph?

Theorem 5.3. If G is disconnected, then $\overline{G}$ is connected.

Proof. Let $a, b \in V_{\overline{G}}$. If a and b are not adjacent in G, then they are in $\overline{G}$. If $ab \in E_G$, choose a vertex c in G belonging to a component different than that containing a and b. Such a vertex c exists if G is disconnected. Then $ac \notin E_G$ and $cb \notin E_G$, and they form a path from a to b in $\overline{G}$. $\triangledown$

Definition. A graph G is *self-complementary* when $\overline{G} \simeq G$.

In fact, you have seen a self-complementary graph (where?) i.e., $C_5 \simeq \overline{C}_5$. Another example, P_4 is also self-complementary—a fact which is not hard to sketch and see. Note that Theorem 5.3 requires a self-complementary graph to be connected.

Exercise 5.18. Find one more example of a self-complementary graph. If it helps, see the next exercise first.

Exercise* 5.19. Prove that a self-complementary graph with n vertices must have exactly $\frac{n(n-1)}{4}$ edges and consequently, $n \bmod 4 \leq 1$.

5.1.3 Matrices

If M denotes an arbitrary matrix, we shall use the notation $[M]_{ij}$ to refer to the (i,j) entry in M, i.e., the entry in the ith row and jth column of M.

Definition. Suppose that $V_G = \{v_1, v_2, \ldots, v_n\}$. The *adjacency matrix* of the graph G is the $n \times n$ matrix A given by $[A]_{ij} = 1$ if $v_i v_j \in E_G$, otherwise $[A]_{ij} = 0$.

For example, the adjacency matrix of C_4—with the standard edges $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$—is given by

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Question. Can non-isomorphic graphs have the same adjacency matrix?

Exercise 5.20. Describe the adjacency matrices of the graphs K_n, $K_{m,n}$, P_n, and C_n.

Exercise* 5.21. Let A denote the adjacency matrix of a graph G with vertices $v_1, v_2, \ldots, v_n$. Prove that $[A^2]_{ii} = \deg(v_i)$.

Definition. A *permutation matrix* is a square matrix obtained from the identity matrix by reordering its rows.

For example, the matrix

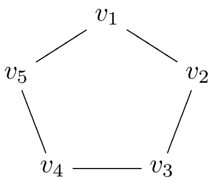
$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{5.2}$$

is a permutation matrix, which we may obtain from the identity matrix by permuting its rows in the order of $(1, 3, 5, 2, 4)$. A known fact is that every permutation matrix P belongs to the family of orthogonal matrices, as we have $P^{-1} = P^T$.

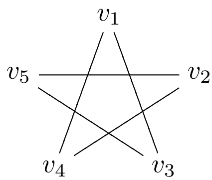
Since isomorphic graphs are essentially relabeling of the vertices, we can expect that their adjacency matrices are related to each other via some permutation matrix. The following theorem, though not quite useful, describes this relation in an elegant manner.

Theorem 5.4. Suppose that A and B are the adjacency matrices of the graphs G and H, respectively. Then $G \simeq H$ if and only if $B = PAP^T$ for some permutation matrix P.

Example. We look again at the fact that $C_5 \simeq \overline{C}_5$, this time with the particular vertex labeling as follows.



$$C_5 \quad \simeq \quad \overline{C}_5$$



An obvious bijection is one which permutes the vertex indices $(1, 2, 3, 4, 5)$ to $(1, 3, 5, 2, 4)$, thus the permutation matrix P given in (5.2). Indeed, the corresponding adjacency matrices A and $\overline{A}$, respectively, satisfy the relation $PAP^T = \overline{A}$, i.e.,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Definition. Suppose that $V_G = \{v_1, v_2, \ldots, v_n\}$ and $E_G = \{e_1, e_2, \ldots, e_m\}$. Then the *incidence matrix* of the graph G is the $n \times m$ matrix Z given by $[Z]_{ij} = 1$ if $v_i \in e_j$, otherwise $[Z]_{ij} = 0$.

For example, the incidence matrix of the path P_4, with standard set of edges $E = \{v_1v_2, v_2v_3, v_3v_4\}$, is the following 4×3 matrix Z.

$$Z = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise 5.22. Describe the incidence matrices of the graphs K_n, $K_{m,n}$, P_n, and Z_n.

Exercise 5.23. Convert the following incidence matrices to adjacency matrices.

$$\text{a) } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{b) } \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{c) } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \text{d) } \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Exercise 5.24. Amira has correctly computed the adjacency matrix of an unlabeled graph to be

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Elias, using a labeling of his own, is to find the incidence matrix. Which one of the following matrices is supposedly Elias's correct answer?

a) $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Definition. Let G be a graph with vertices $v_1, v_2, \ldots, v_n$. The *degree matrix* of G is the $n \times n$ diagonal matrix D given by $[D]_{ii} = \deg(v_i)$.

Question. Can you describe the degree matrix of the Peterson graph?

Theorem 5.5. Suppose that the adjacency matrix A and incidence matrix Z have been given for the same graph G. Then $ZZ^T = A + D$, where D is the degree matrix of G.

Proof. For $i \neq j$, we have $[ZZ^T]_{ij} = \sum_{k \geq 1} [Z]_{ik}[Z]_{jk}$. If $v_i v_j \in E$, then there is exactly one value of k for which $[Z]_{ik} = [Z]_{jk} = 1$. In that case, both ZZ^T and $A + D$ have 1 in their (i, j) entries. If $v_i v_j \notin E$, no such k exists, and that entry will be 0 in both. Lastly, if $i = j$, then $[ZZ^T]_{ii} = \sum_{k \geq 1} [Z]_{ik}$, counting the number of vertices adjacent to v_i, which agrees with the ith diagonal entry in $A + D$. $\triangledown$

5.2 Introduction to Trees

Definition. A *tree* is a connected graph which contains no cycles. In general, a graph which contains no cycles is called *acyclic*.

For example, K_4 contains the cycle C_4 (as well as C_3), hence K_4 is not a tree. On the other hand, P_4 is a tree because no subgraph of P_4 is a cycle.

Test 5.25. Which one of these four graphs is a tree?

a) K_9
b) $K_{9,1}$
c) $K_{9,9}$
d) C_9

Exercise 5.26. Determine exactly when each of the graphs K_n, $K_{m,n}$, P_n, and C_n, is a tree.

Question. Is the Peterson graph a tree?

Exercise 5.27. Draw all non-isomorphic trees with no more than 6 vertices.

Theorem 5.6. Let G be a connected graph. The following statements are equivalent one to another.

1) The graph G is acyclic.

2) Every edge in G is a bridge.

3) The size of G is determined by $|E_G| = |V_G| - 1$.

4) There is a unique path between any two vertices in G.

Proof. Suppose that from a to b we have two distinct paths. The union of the two paths make a cycle. A cycle contains no bridge, for removing an edge results in a path through all existing vertices. Now removing a non-bridge edge, if any, keeps the graph connected, hence $|E| > |V| - 1$ by Exercise 5.14. We have established $\neg(4) \to \neg(1) \to \neg(2) \to \neg(3)$, and we leave it to you now to complete the proof of $\neg(3) \to \neg(4)$. $\triangledown$

Exercise 5.28. Prove that adding an edge to a tree will produce a cycle.

Definition. In a graph, a vertex of degree one is called a *leaf.*

Exercise 5.29. Show that any one of the special graphs K_n, $K_{m,n}$, P_n, and C_n, may have a leaf if and only if it is a tree.

Theorem 5.7. Every tree has a leaf. In fact, a tree has $2 + \sum_{i\geq 3}(i-2)n_i$ leaves, where n_i denotes the number of vertices of degree i.

Proof. Let $d_1, d_2, \ldots, d_n$ be the degree sequence of a tree. Being connected, we have $d_n \geq 1$. With $n-1$ edges, Euler's theorem gives $\sum d_i = 2n-2$. The number of leaves is minimum in the case $2, 2, \ldots, 2, 1, 1$. A vertex of degree $i \geq 3$ reduces this number of twos, while increases the number of leaves—exactly $i-2$ of them. This yields the claimed formula. $\triangledown$

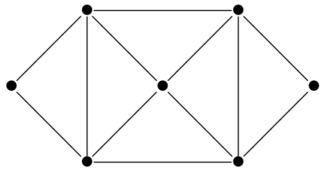
Exercise* 5.30. So the degree sequence $d_1, d_2, \ldots, d_n$ of a tree, with $d_n > 0$, always sum to $2n-2$. Conversely, show that every degree sequence with this property is graphical and can be given by a tree.

Definition. A *spanning tree* of a graph G is a tree $T \subseteq G$ with $V_T = V_G$.

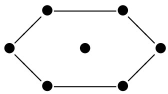
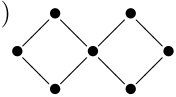
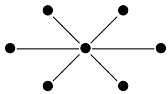
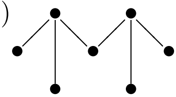
For example, P_4 is a spanning tree of both C_4 and K_4. Alternately, a spanning tree for K_4 can be $K_{1,3}$.

Question. Can you find a spanning tree for the Peterson graph in the form of a path?

Test 5.31. Which one of the following graphs is a spanning tree of the graph given in (5.3) below?



(5.3)

a)  b)  c)  d) 

The following algorithm can be used to generate a spanning tree of a given graph, if connected. If G is disconnected, of course, G cannot have a spanning tree.

Algorithm 5.8 (Depth-First Search)**.** Given a graph G, we determine whether or not G is connected, in which case, we produce a spanning tree of G. We shall label the vertices of G as $v_1, v_2, v_3, \ldots$, based upon the order of traversal as follows.

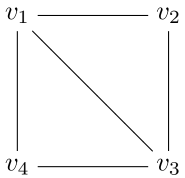
1) Start with an arbitrary vertex and label this vertex v_1.
2) Move to any vertex adjacent to the current selection which has not been labeled. If no such vertex exists, backtrack to a previously visited vertex.
3) Repeat the previous step until no more move is possible.
4) If all vertices of G have been labeled, then G is connected and this traversal generates a spanning tree of G.

Observation. The algorithm works simply because G is connected if and only if every other vertex will eventually be traversed, regardless the choice of v_1. That the resulting graph is a tree is due to the fact that it takes exactly one new edge to add one new vertex during the entire procedure. $\triangledown$

In general, a graph can have many spanning trees, unless G is already a tree, in which case G has no spanning tree other than G itself. The next theorem provides an algorithm to compute the number of spanning trees of a graph with labeled vertices. The proof, first given by Kirchoff, can be found in many graph theory texts.

Theorem 5.9 (Matrix Tree Theorem)**.** Let G be a connected graph with labeled vertices, adjacency matrix A, and degree matrix D. Then any cofactor of the matrix $D - A$ will give the number of spanning trees of G.

Example. The graph G is given, together with the associated matrix $D-A$.



$$G \quad \rightarrow \quad D - A = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

Recall that the cofactor $C_{i,j}$ of a square matrix M is the determinant of the matrix obtained from M by removing the ith row and jth column, e.g.,

$$C_{3,1} = \det \begin{bmatrix} -1 & -1 & -1 \\ 2 & -1 & 0 \\ 0 & -1 & 2 \end{bmatrix}$$

We compute this determinant to conclude that there are 8 spanning trees for the graph G.

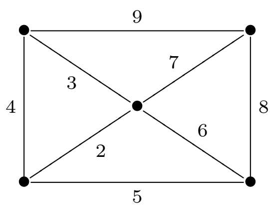
Exercise 5.32. Repeat the example using P_4, K_4, $K_{2,3}$, and C_5.

Exercise* 5.33. There are n^{n-2} different trees with vertices $v_1, v_2, \ldots, v_n$, if $n \geq 2$. Prove this claim using the matrix tree theorem by showing that n^{n-2} is the number of spanning trees of a labeled K_n.

Suppose the vertices represent towns in a state whose government wishes to build railroads connecting all of them. To save cost, it would be wise to choose not only a spanning tree, but one whose edges sum to the least possible kilometers. This is one application of a minimal spanning tree.

Definition. A graph is *weighted* if every edge is associated with a numerical value, called *weight*. For us, weighted graphs are allowed only non-negative values. A *minimal spanning tree* of a weighted graph is a spanning tree with the least total weight.

Question. What is the least possible total weight for a spanning tree of the weighted graph (5.4) below?



(5.4)

In order to present the algorithm by which we can find a minimal spanning tree of a weighted graph, we first need a definition.

Definition. For every vertex $v \in V$, let $N(v) = \{w \in V \mid vw \in E\}$, i.e., the set of all vertices which are adjacent to v. Now if $S \subseteq V$, we define the *neighborhood* of S to be the set $N(S) = \bigcup_{v \in S} N(v)$.

Algorithm 5.10 (Prim)**.** Given a connected weighted graph, we produce a minimal spanning tree as follows.

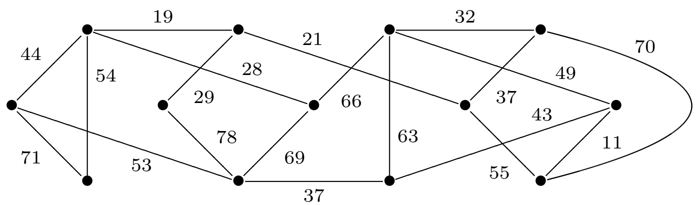
1) Select an arbitrary vertex. Let S be the set of vertices which have already been selected.

2) Add to S one more vertex from $N(S)$ such that the corresponding edge is of least weight.

3) Repeat until $|S| = |V|$.

Observation. It is clear that the resulting subgraph is a tree since we would have selected exactly $|V|$ vertices and $|V| - 1$ edges, keeping the subgraph connected at each selection. Although minimal spanning tree may not be unique, it can be shown that this algorithm always yields the least possible total weight. $\bigtriangledown$

Exercise 5.34. Apply Prim's algorithm to the weighted graph below.



5.3 Walks

Definition. A *walk* is simply a sequence of continuous edges, i.e., of the form $v_1v_2, v_2v_3, \ldots, v_nv_{n+1}$. In this case, it is a walk of *length* n from v_1 to v_{n+1}. If it happens that $v_{n+1} = v_1$, we have what we call a *closed walk* of length n. We loosely treat a walk as a graph containing these edges and the vertices therein.

In particular, a path P_n can be considered as a walk of length $n - 1$, while C_n a closed walk of length n. In a walk, the edges involved are not assumed distinct, and neither are the vertices—unless of course, the walk is a path.

Theorem 5.11. If G is a closed walk of odd length, then G contains a cycle of odd length.

Proof. A closed walk of length three is none other than C_3, so the claim is true. We proceed by induction, assuming the theorem has been proved for all lenghts less than n. If G is the walk $v_1v_2, v_2v_3, \ldots, v_nv_1$ with no repeated vertex, then $G \simeq C_n$ and we are done. Suppose now $v_i = v_{i+j}$. Then G is really the union of two closed walks: the one from v_i to v_{i+j}, and the walk from v_1 to v_i joined by that from v_{i+j} to v_1. One of the two must have an odd length, since their sum is, and which is clearly less than n, hence it contains a cycle of odd length. $\triangledown$

Theorem 5.12. Let A denote the adjacency matrix of a graph G with vertices $v_1, v_2, \ldots, v_n$. The number of walks of length k from v_i to v_j is then given by $[A^k]_{ij}$.

Proof. It is trivial that $[A]_{ij} = 1$ if and only if there is a walk of length one from v_i to v_j. We proceed by induction. Since a walk of length $k+1$ from v_i to v_j consists of a walk of length k from v_i to an intermediate vertex v_m, which must be adjacent to v_k, then the total number of such walks is just

$$\sum_{m=1}^{n} [A^k]_{im}\,[A]_{mj} = [A^{k+1}]_{ij}$$

and the induction is complete. $\triangledown$

In particular, a closed walk of length 3 is a triangle. Since the vertices of a labeled triangle can be permuted in 6 different ways, we conclude that a labeled graph given by its adjacency matrix A contains

$$\frac{1}{6}\sum_{i\geq 1} [A^3]_{ii}$$

triangles. In matrix algebra, the sum of diagonal entries is called the *trace*. Thus, one-sixth of the trace of A^3 counts the number of triangles.

Question. How many triangles do you see in the graph given in (5.3)?

Exercise 5.35. Find the number of triangles contained in the graphs K_4, $K_{2,2}$, $K_{2,3}$, and K_5.

Question. Can you find an easy formula for the number of triangles contained in K_n?

5.3.1 Distances

Definition. The *distance* between two vertices a and b, denoted by $d(a,b)$, is the length of the shortest walk from a to b, if it exists, or else let $d(a,b)=\infty$.

For example, if a and b are adjacent, then $d(a,b)=1$. Also, by definition, we have $d(a,a)=0$. Note that the shortest walk is necessarily a path.

Definition. The *diameter* of a graph G, denoted by $d(G)$, is the largest possible distance between two vertices in G.

Hence, for example, $d(G)=1$ if and only if G is complete. Note also that $d(G)=\infty$ if and only if G is disconnected.

Question. What is the diameter of the Peterson graph?

Test 5.36. Which one of these four graphs has the largest diameter?

a) K_{99}
b) $K_{99,99}$
c) P_{99}
d) C_{99}

Exercise 5.37. Find the diameters of the graphs K_n, $K_{m,n}$, P_n, and C_n.

Theorem 5.13. If $d(G)\geq 3$ then $d(\overline{G})\leq 3$.

Proof. Assume $d(G)\geq 3$ and take two vertices v and w. We will show that $d(v,w)\leq 3$ in $\overline{G}$.

We know there are a and b for which $d(a,b)\geq 3$ in G. It is then false that both $av, vb\in E_G$ or both $aw, wb\in E_G$. Hence, $E_{\overline{G}}$ must contain either av and aw, in which case $d(v,w)\leq 2$ in $\overline{G}$; or vb and wb, again we are done; or av and wb; or else vb and aw. In these last two cases, with the fact that $ab\in E_{\overline{G}}$, we would have $d(v,w)\leq 3$ in $\overline{G}$. ▽

Exercise* 5.38. If G is self-complementary, prove that $d(G)=2$ or 3.

Definition. Suppose that $V_G=\{v_1,v_2,\ldots,v_n\}$. The *distance matrix* of the graph G is the $n\times n$ matrix D given by $[D]_{ij}=d(v_i,v_j)$.

For example, the distance matrix of P_4, labeled in the standard way, is given by

$$D=\begin{bmatrix}0&1&2&3\\1&0&1&2\\2&1&0&1\\3&2&1&0\end{bmatrix}$$

Question. Can non-isomorphic graphs have identical distance matrices?

Exercise 5.39. Describe the distance matrices of the graphs K_n, $K_{m,n}$, P_n, and C_n.

Exercise 5.40. Convert the incidence matrices given in Exercise 5.23 to distance matrices.

Exercise 5.41. Consider again the adjacency matrix A given in Exercise 5.24. Which one of the following is the corresponding distance matrix?

$$\text{a)}\begin{bmatrix}0&1&1&1\\1&0&2&1\\1&2&0&2\\1&1&2&0\end{bmatrix}\quad \text{b)}\begin{bmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{bmatrix}\quad \text{c)}\begin{bmatrix}0&1&1&1\\1&0&2&2\\1&2&0&2\\1&2&2&0\end{bmatrix}\quad \text{d)}\begin{bmatrix}0&1&1&1\\1&0&2&1\\1&2&0&3\\1&1&3&0\end{bmatrix}$$

With the help of a computer, it is not hard to convert an adjacency matrix to the distance matrix. The following algorithm shows that what it takes is to compute the powers of A.

Algorithm 5.14 (From A to D)**.** Given the $n \times n$ adjacency matrix A, we are able to retrieve the distance matrix D following these steps.

1) Compute the matrices $A, A^2, A^3, \dots, A^n$.
2) Set $[D]_{ii} = 0$, and for $i \neq j$ let $[D]_{ij} = k$, the least exponent for which $[A^k]_{ij} \neq 0$. If no such k exists, set $[D]_{ij} = \infty$.

Observation. Theorem 5.12 says that $[A^k]_{ij}$ is the number of walks from i to j of length k. The first nonzero k is therefore the distance, and if there is no walk from i to j of length n or less, then there is no such walk. $\triangledown$

Definition. If G is a weighted graph, we redefine the distance $d(a, b)$ to be the least total weight of all possible walks from a to b.

Question. What is the largest possible distance between two vertices in the weighted graph given in (5.4) earlier?

The next algorithm, which we will present without proof, computes distances in a weighted graph. It reminds us of Prim's algorithm for spanning trees, in the way that it repeatedly search for the least next weight within an increasingly bigger neighborhood.

Algorithm 5.15 (Dijkstra)**.** We compute $d(a, b)$ in a weighted graph and find the shortest path (least total weight) from a to b.

1) We denote by S the set of vertices s which have already been labeled by (v_s, W_s), where v_s is a vertex and W_s is an integer. Initially, S contains only the vertex a, which we label $(-, 0)$.

2) For each vertex $x \in N(S)$, say adjacent to $y \in S$ with weight W, calculate the number $W_x = W + W_y$. Choose the least of such numbers, and label the corresponding vertex x, or vertices if not unique, by (y, W_x).

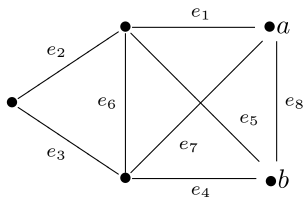
3) Repeat until b is labeled, in which case $d(a,b) = W_b$. Moreover, the shortest path from a to b can be found by backtracking the first components in the labels.

Exercise 5.42. Visit again the weighted graph given in Exercise 5.34 and apply Dijkstra's algorithm to compute the distance from east to west, i.e., from the far-right vertex to the far-left vertex.

5.3.2 Euler Circuits

Definition. An *Euler walk* in a connected graph G is a walk through all the edges of G without repeating any of them. If an Euler walk is closed, we shall call it an *Euler circuit.*

As an example, we show below a graph which has an Euler walk from a to b by following the labeled edges $e_1, e_2, \ldots, e_8$, in this order.



Question. Does the Peterson graph have an Euler walk?

Theorem 5.16. A connected graph has an Euler walk from a to $b \neq a$ if and only if a and b are the only vertices whose degrees are odd. The graph has an Euler circuit if and only if all vertices have even degrees.

Proof. Consider a vertex v with $\deg(v) = d$. At some point during the walk, we will run into v and out via another edge. If $d > 2$, this process will repeat, for as long as there are untrodden edges containing v. This shows the necessity that d be even, unless v is our starting point, or last destination, in which case d may be odd.

To prove sufficiency, assume first that every vertex in a graph G has an even degree. Consider a path P of maximum length from v to w. Since $\deg(w)$ is at least two, w is adjacent to another vertex already contained in P, else we could extend P to a longer path. Hence G contains a cycle

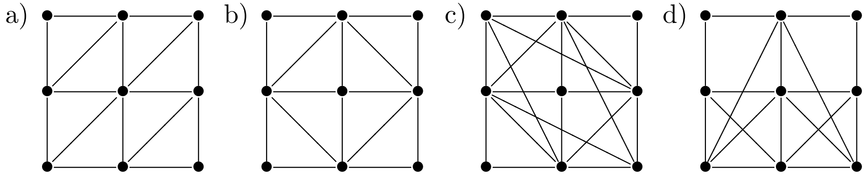
C. Since every vertex in C has an even degree, so does the subgraph whose edges are in $G - C$. Repeating the argument, we see that the edges in G can be thought of as the union of cycles none of which is disjoint from the rest.

We finish off by induction. One cycle is itself an Euler circuit. Assume that the union of n such cycles has an Euler circuit, call it E. With one more cycle C, which meets E at a vertex x, we have an Euler circuit for $E \cup C$ by starting at x, circuiting around E back to x, and cycling around C back to x.

Lastly, if $\deg(a)$ and $\deg(b)$ are the only odd degrees in G, we add one more edge, i.e., ab into G (perhaps making G a multigraph) so that every vertex now has an even degree. We have shown that an Euler circuit exists for this extended graph. Hence, without this extra edge, we could Euler walk from a and terminate at b. $\triangledown$

Exercise* 5.43. Prove that Theorem 5.16 holds for multigraphs as well.

Exercise 5.44. With the help of Theorem 5.16, find an Euler walk or Euler circuit, if any, in each graph given below.



Test 5.45. Which one of these four graphs has an Euler walk but not Euler circuit?

a) K_{99}
b) K_{100}
c) $K_{100,2}$
d) $K_{99,2}$

Exercise 5.46. Determine exactly when each of the graphs K_n, $K_{m,n}$, P_n, and C_n, may have an Euler walk or an Euler circuit.

Amira is a traditional postal carrier who walks through every street within her district of responsibility to deliver the mail. Having studied graph theory, she searches for a closed route which involves the least number of repeated streets or, if possible, none at all.

The *Chinese postman problem* asks for the shortest closed walk going through every edge in a graph, possibly weighted. If any, an Euler circuit would certainly give the optimal solution, or else such a walk would have to repeat one or more edges. The algorithm for solving this problem is next.

Algorithm 5.17 (Chinese Postman Problem)**.** The shortest closed walk covering every edge in the graph, with or without weights, is obtained following these steps.

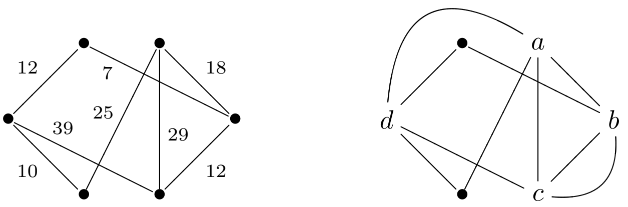
1) Identify all vertices which have odd degrees. By Euler's theorem, their number is even.

2) Pair up these odd vertices $\{a_1, b_1\}, \{a_2, b_2\}, \ldots, \{a_n, b_n\}$ in such a way that $\sum d(a_i, b_i)$ is the least possible.

3) The shortest walk solution is through all the edges of G, plus the shortest paths from a_i to b_i, where $1 \le i \le n$.

Observation. Any closed walk with no repeated edges will visit each vertex an even number of times, so a vertex with odd degree necessitates a repetition. One would think that these extra walks might well be paths, in order to optimize the length, and that each path should start and end at an odd degree, to make good use of their oddness. These claims, though intuitively acceptable, would need rigorous justifications. ▽

Example. We solve the Chinese postman problem for the weighted graph shown on the left in (5.5). After a brief examination, we find four vertices of odd degree which we label a, b, c, d, in the figure on the right.



(5.5)

The imaginary edges $\{a, d\}$ and $\{b, c\}$ show the optimal pairing, after exhausting the three possibilities, i.e.,

$$\begin{aligned} d(a, b) + d(c, d) &= (18) + (12 + 7 + 12) = 49 \\ d(a, c) + d(b, d) &= (29) + (7 + 12) = 48 \\ d(a, d) + d(b, c) &= (25 + 10) + (12) = 47 \end{aligned}$$

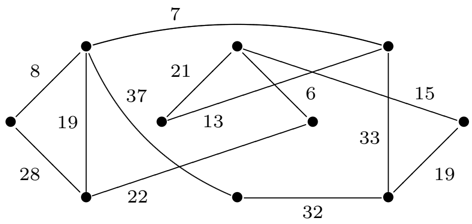
Hence, the minimal closed walk is the Euler circuit on the multigraph, whose weight is the overall total 152 plus the extra edge ad, which is really the path of weight $25 + 10$, and bc, of weight 12. These sum to 199 total weight.

Question. What is the minimal length solution to the above example if the graph were unweighted?

Test 5.47. In solving the Chinese postman problem, suppose that there are 6 vertices of odd degree. How many different pairings are possible?

a) 6
b) 12
c) 15
d) 30

Exercise 5.48. Solve the Chinese postman problem for the graph below.



5.3.3 Hamilton Cycles

Definition. A *Hamilton cycle* in a graph G is a cycle $C_n \subseteq G$, where $n = |V_G|$. A *Hamilton graph* is one which contains a Hamilton cycle.

In other words, a Hamilton cycle is a closed walk through all the vertices in G without repeating any of them except, of course, the starting vertex. Note that a Hamilton graph is necessarily connected.

Question. Why is a Hamilton graph never a tree, nor vice versa?

Test 5.49. Which one of these four is *not* a Hamilton graph?

a) K_{99}
b) $K_{99,99}$
c) $K_{99,100}$
d) C_{99}

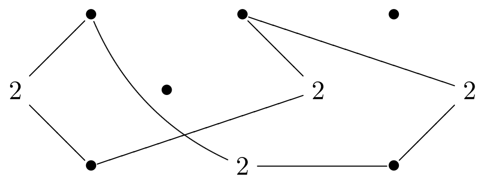
Exercise 5.50. Determine exactly when each of the graphs K_n, $K_{m,n}$, P_n, and C_n, is a Hamilton graph.

We know no effective algorithms that determine whether a given graph contains a Hamilton cycle, let alone produce one if any. The next theorem states some, rather weak necessary conditions for a Hamilton graph, followed by another theorem on a sufficient condition.

Theorem 5.18. If G is a Hamilton graph, then G contains no leaves, no bridges, and no cut vertices. A *cut vertex* is one that disconnects the graph when removed.

Proof. A Hamilton cycle, or any cycle, is 2-regular. This necessitates every vertex in G to have degree at least two. Now a bridge is the only connection between two components. So a closed walk through both components must cross the bridge twice, hence not a cycle. Similarly, a closed walk through a cut vertex must repeat the vertex, hence not a cycle. $\bigtriangledown$

The converse of the theorem does not hold. The graph given in Exercise 5.48, for example, has no Hamilton cycle. If it did, the vertices of degree two must yield both edges, creating an impossible subcycle shown below.



Exercise* 5.51. Prove that the Peterson graph does not contain a Hamilton cycle.

Theorem 5.19. If every vertex in a connected graph G has degree at least $|V|/2$, then G is a Hamilton graph.

Proof. Let $|V| = n$ and P be a path of maximum length in G, given by $v_1v_2, v_2v_3, \ldots, v_{k-1}v_k$. Having maximum length means that the vertex v_1 is not adjacent to any other outside P, and similarly for v_k. Since $\deg(v_1) \geq n/2$, there are at least this many vertices in P adjacent to v_1, and similarly to v_k. By the pigeonhole principle, we can find v_j, with $2 \leq j \leq k$, such that both $v_1v_j, v_{j-1}v_k \in E$. This gives us a cycle from v_1 to $v_{j-1}, v_{j-1}v_k$, backward to v_j, v_jv_1. Then any additional vertex connected to this cycle would contradict the maximality of the length of P, hence $k = n$. $\bigtriangledown$

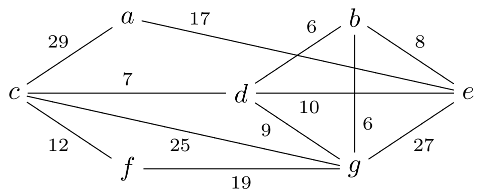
For example, we may use Theorem 5.19 to show that a complete graph with at least 3 vertices has a Hamilton cycle. But we already know that.

Exercise 5.52. Prove that the complement of the Peterson graph is a Hamilton graph, and so is the complement of C_n, if $n \geq 5$.

Now Elias is a traveling salesman who is planning to visit several major cities for business purposes. Having studied graph theory, Elias searches for a cycling flights through all these cities and back to his hometown, which will cost him the least possible airfares.

Thus the *traveling salesman problem* asks for a Hamilton cycle of least length in a weighted graph. One way to solve the problem is to try out all possible Hamilton cycles and select the one with least total weight, for there is no decisive algorithm that works in all cases.

Example. Let us solve the traveling salesman problem for the following weighted graph.



(5.6)

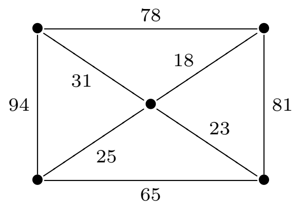
A cycle takes two edges from each vertex. Due to a and f, of degree two, any Hamilton cycle must contain the path $eacfg$, of weight $17+29+12+19 = 77$. To complete the cycle, we continue with either the path $gbde$, of weight $6+6+10 = 22$, or $gdbe$, of weigth $9+6+8 = 23$. The choice is now obvious: the Hamilton cycle $eacfgbde$, of total weight 99.

Exercise 5.53. Follow the above example and solve the traveling salesman problem with the weighted graph given in (5.5) earlier.

Question. What is the traveling salesman solution to the weighted graph given in (5.4)?

Exercise* 5.54. Find the traveling salesman solution, if any, using the weighted graph given in Exercise 5.34.

Again, there is no generalized technique for solving the traveling salesman problem. And by the way, the shortest closed walk going through every vertex is not always given by a cycle. In the following hypothetical example, taking a walk anywhere is always cheaper via the center vertex!



(5.7)

Exercise 5.55. What will be the least possible total weight for a closed walk through every vertex in the weighted graph given in (5.7)?

5.4 Coloring

One important problem about graphs concerns the idea of vertex coloring. What we mean by vertex coloring is assigning a color to every vertex in such a way that adjacent vertices have different colors. An obvious challenge is then to do this task using as few colors as possible. We discuss first the case where two colors suffice.

5.4.1 Bipartite Graphs

For the next definition, we first introduce the new set notation $X \sqcup Y$, which really stands for the union $X \cup Y$ together with the additional assumption that X and Y are disjoint. We say that $X \sqcup Y$ is the *disjoint union* of X and Y, and that X and Y are the *bipartition subsets* of $X \sqcup Y$.

Definition. A graph G is *bipartite* if $V_G = X \sqcup Y$, such that two vertices are adjacent only if exactly one of them belongs to X (and the other to Y).

Note that if G is disconnected then G is bipartite if and only if each component is bipartite. Hence, in discussing bipartite graphs, we assume that G is a connected graph, unless otherwise stated.

Question. Is the Peterson graph bipartite?

Test 5.56. Which one of these four graphs is *not* bipartite?

a) K_9
b) $K_{8,8}$
c) P_7
d) C_6

The complete bipartite graphs $K_{m,n}$ are but a special subfamily of bipartite graphs, having the property that two vertices are adjacent if and only if they do not belong together in the bipartition subsets.

Theorem 5.20. Let G be any non-trivial graph. The following statements are equivalent one to another.

1) The graph G is bipartite.

2) Two colors are sufficient to color the graph G.

3) There is a complete bipartite graph which contains G.

4) There is no cycle of odd length in G.

Proof. By definition, it is clear that a bipartite graph is a subgraph of some $K_{m,n}$. Moreover, in the case $V = X \sqcup Y$, it suffices to color the vertices in X black and Y white. Hence, if a cycle $v_1v_2, v_2v_3, \ldots, v_nv_1$ is a subgraph, then n is even, or else v_1 and v_n would be of the same color.

To complete the proof, we assume that G contains no cycle of odd length, and we will show G is bipartite. Fix a vertex v and let $X = \{w \in V_G \mid d(v, w)$ is even$\}$ and $Y = \{w \in V_G \mid d(v, w)$ is odd$\}$. It is clear that $V_G = X \sqcup Y$. Moreover, if $a, b \in X$ then there is a walk of even length from v to each of a and b. Then a and b cannot be adjacent, otherwise the closed walk from v to a, then ab, from b back to v, would have an odd length, and that is not allowed by Theorem 5.11. Similarly, if $a, b \in Y$, we see that $ab \notin E$. Thus G is bipartite with these bipartition subsets X and Y. $\triangledown$

Question. Why is a tree always a bipartite graph?

Exercise 5.57. Determine exactly when each of the graphs K_n, $K_{m,n}$, P_n, and C_n, is bipartite.

Exercise* 5.58. Let G be a bipartite graph with n vertices. Prove that $|E_G| \leq n^2/4$, where equality holds if and only if $G \simeq K_{n/2,n/2}$.

Algorithm 5.21 (Bipartite Graph Coloring)**.** We may determine whether G is bipartite, and if so we find the bipartition subsets, as follows.

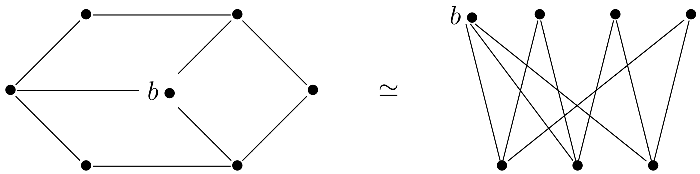
1) Select an arbitrary vertex and color it black. Let S denote the set of vertices which have already been colored.

2) For each vertex in $N(S)$ color it white or black, such that adjacent vertices have distinct colors. If this is not possible, then G is not bipartite.

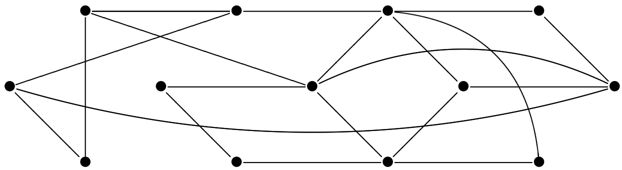
3) Repeat until $|S| = |V_G|$. In the end, the vertices are bipartitioned according to their color.

Observation. If G is bipartite, two colors suffice. The color of a vertex is then completely determined by the color of its adjacent vertex, regardless where we start. If G is not bipartite, such procedure is doomed to fail. $\triangledown$

Example. We apply this coloring algorithm to the graph shown on the left. An initial vertex b is chosen black, and we successfully complete the coloring of all seven vertices with black and white. The redrawing on the right shows more apparently that the graph is indeed bipartite.



Exercise 5.59. Apply the coloring algorithm to see if the given graph below is bipartite.



Suppose that Elias runs an online dating service. In terms of graph theory, the bipartition subsets are men and women who are registered members of this service. Agreeably, a match is when a man and a woman have a certain number of common interests in different categories. Knowing that it is not ethical to match two men to the same woman, or vice versa, Elias consults his graph theory text to find the maximum set of matches for his collection of data.

Definition. A *matching* in a bipartite graph G is a subgraph $M \subseteq G$ in which the edges are mutually disjoint. If $V_G = X \sqcup Y$, then a matching M is called *complete* for X if M contains X. And if, in addition, $|X| = |Y|$ then any complete matching (for X or Y) is a *perfect matching*.

Note that if an edge ab is treated like a relation from X to Y, then a matching is just like a one-to-one function, which is complete when the domain is all of X. A perfect matching is therefore like a bijection, which requires that $|X| = |Y|$ and in which case its inverse is also a bijection, from Y to X.

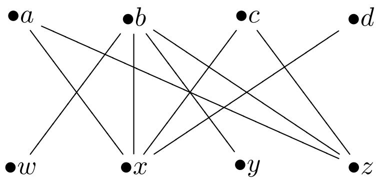
Suppose that G is bipartite with bipartition $V_G = X \sqcup Y$ and a matching M. Let the path $P = \{v_1v_2, v_2v_3, \ldots, v_{2n-1}v_{2n}\}$ be such that the edges $v_kv_{k+1} \in M$ if and only if k is even. If, in addition, both $v_1, v_{2n} \notin M$, then replacing the edges in $M \cap P$ by those in $P - M$ will produce a new matching M' which contains one more vertex from each X and Y. Such "alternating" path is the key to proving the next theorem.

Theorem 5.22 (Hall's Theorem)**.** Let G be bipartite with $V_G = X \sqcup Y$. Then G has a complete matching for X if and only if $|S| \leq |N(S)|$ for every subset $S \subseteq X$.

Proof. Necessity is obvious for if $S \subseteq X$ and G has a complete matching M, then in M, every vertex in S is adjacent to its own vertex in Y, thus $|N(S)| \geq |S|$. Now assume, for sufficiency, $|S| \leq |N(S)|$ for every subset $S \subseteq X$. Assume also, by induction, that we have a matching M contaning all of X except one vertex $v \in X$. We will produce an alternating path from v, with respect to M, which will complete this matching for X.

Since $N(\{v\})$ is not empty, we can find $w_1 \in Y$ which is adjacent to v. If $w_1 \notin M$ then vw_1 is such path, we are done. Else, there is an edge $v_1w_1 \in M$. Since $|N(\{v, v_1\})| \geq 2$, we have w_2, adjacent to either v or v_1. If $w_2 \notin M$, again we have an alternating path from v to w_2, so assume there is another edge $v_2w_2 \in M$. Continuing in this way, seeing that $|N(\{v, v_1, v_2, \dots v_k\})| > k$ in each step, we will exhaust the vertices in $Y \cap M$, forcing a vertex $w \notin M$ to which there is an alternating path from v. $\bigtriangledown$

For example, in the bipartite graph given in (5.8) below, a complete matching is not possible since we have $|S| > |N(S)|$ for the set $S = \{a, c, d\}$.



(5.8)

Theorem 5.23. Suppose that G is bipartite with $V_G = X \sqcup Y$. If G is regular, then $|X| = |Y|$ and G has a perfect matching.

Proof. Assume that $|X| = n$ and that G is d-regular. Then G has exactly dn edges. Similarly, if $|Y| = m$ then $|E| = dm$. Hence $m = n$. Furthermore, any set S of k vertices in X corresponds to dk edges, hence to k vertices in Y. We have $|N(S)| = |S|$ and a perfect matching by Hall's theorem. $\bigtriangledown$

Exercise 5.60. Draw all regular bipartite graphs with up to 6 vertices.

Exercise* 5.61. Prove that any 2-regular bipartite graph is a cycle of even length, and conversely.

5.4.2 Chromatic Numbers

Definition. The *chromatic number* $\chi(G)$ of a graph G is the least number of colors needed to color the vertices of G, such that adjacent vertices have distinct colors. For example, we have seen that $\chi(G) = 2$ if and only if G is a bipartite graph.

Note that if $H \subseteq G$, then $\chi(H) \leq \chi(G)$. In particular, if G is disconnected then $\chi(G)$ is simply the largest chromatic number among the components of G, so we may well assume henceforth that G is connected where $\chi(G)$ is concerned.

Question. What is the chromatic number of the Peterson graph?

Test 5.62. Which one of these graphs has the largest chromatic number?
a) P_{99}
b) C_{99}
c) P_{100}
d) C_{100}

Exercise 5.63. Determine the chromatic numbers of the graphs K_n, $K_{m,n}$, P_n, and C_n.

Theorem 5.24. Let $\chi(G) = k$. Then the graph G has a vertex v such that $\deg(v) \geq k - 1$, and at least k such vertices.

Proof. Note that removing an edge off G will result in $\chi(G) \leq k$. Let G' be the subgraph of G obtained upon removing as many edges as possible while maintaining $\chi(G') = k$. Hence, removing an edge off G' will make $\chi(G') \leq k - 1$.

It is clear that G' has at least k vertices. We claim that $\deg(v) \geq k - 1$ holds in G', hence in G as well, for each vertex $v \in V_{G'}$. If this were not so, let $\deg(v) \leq k - 2$ in G'. Not counting v in, G' can be colored with $k - 1$ colors or less. So if v has only $k - 2$ adjacent vertices in G', one of the $k - 1$ colors can be assigned to v and we get $\chi(G') \leq k - 1$, a contradiction. $\triangledown$

As a consequence of the preceding theorem, we have a bound for the chromatic number with respect to the degrees of vertices, i.e.,

$$\chi(G) \leq \Delta(G) + 1$$

where $\Delta(G)$ is the largest degree of a vertex in G.

Exercise 5.64. Find two families among the special graphs K_n, $K_{m,n}$, P_n, and C_n, for which $\chi(G) = \Delta(G) + 1$. It can be proved that these are the *only* two classes for which equality holds.

Amira manages the schedule of baseball practice for the Ammon Little League. Practice sessions are to be held at various schools spread throughout town. To avoid family conflict, if there are siblings in two different teams, the two teams should be assigned a different practice day. Wishing to minimize the number of practice days in a week, Amira realizes that the challenge is really to find the chromatic number for her teams.

There are algorithms which generate a coloring solution for G using $\Delta(G)$ colors or less, but there is none so far which can effectively compute $\chi(G)$.

Exercise 5.65. Find the chromatic number for each given graph.
a) The graph given in (5.6).
b) The graph given in (5.5).
c) The graph given in Exercise 5.48.
d) The graph given in Exercise 5.34.

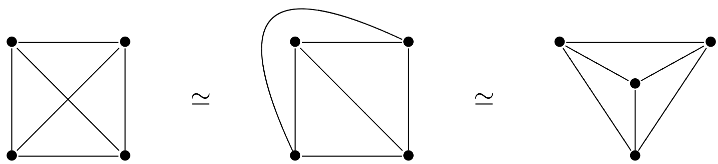
5.4.3 Planar Graphs

Definition. A graph is *planar* if it can be drawn in the plane such that no edges are crossing each other. This particular drawing of a planar graph is called a *plane graph.*

Note that if G is disconnected, then G is planar if and only if each component is planar, hence we may well assume that henceforth G is connected where planarity is concerned.

Question. What is the difference between a planar graph and a plane graph?

For example, although the usual drawing of K_4 involves two edges crossing each other, we may in fact redraw the graph in a way that shows K_4 is planar. We illustrate two possible ways of drawing K_4 as follows.



However, you may try in vain to do the same with the graph $K_{3,3}$, for instance, because $K_{3,3}$ is not a planar graph. Since a subgraph of a planar graph is obviously planar, we see that $K_{m,n}$ is not planar if both $m, n \geq 3$.

Question. Can you *prove* that $K_{3,3}$ is not planar?

Exercise 5.66. Prove that $K_{2,n}$ is planar for all $n \geq 1$.

A plane graph partitions the plane into subsets which are called *regions.* Thus, regions refer to the bounded areas interior to the plane graph, plus one unbounded exterior. For example, the plane graph of K_4 has four regions.

Theorem 5.25 (Euler's Formula)**.** Suppose that a connected plane graph has v vertices, e edges, and r regions. Then $v + r = e + 2$.

Proof. If the graph is a tree, then $e = v - 1$ and $r = 1$ since there is no bounded region. Hence, $v + r - e = 2$ as desired. If not a tree, then the graph can be obtained by adding edges to its spanning tree. Each new edge would add a bounded region, increasing the value of r, as well as e, by one. Thus the equality $v + r = e + 2$ is preserved. $\triangledown$

Exercise* 5.67. Prove that Euler's formula holds for connected plane multigraphs as well.

Theorem 5.26. Let G be a planar graph with $n \geq 3$ vertices. The following results are consequences of Euler's formula.

1) The number of edges in G is at most $3n - 6$.

2) If $n \geq 5$ then G is not a complete graph.

3) There is a vertex in G of degree 5 or less.

4) If G contains no triangles, then G has at most $2n - 4$ edges.

Proof. In the drawing of a plane graph, we note that the number of edges is maximized when every region is the interior of a triangle. (With higher polygons, a diagonal edge can be added while keeping G planar.) Since each edge borders two regions, we have $2|E| = 3r$, where r is the number of regions. Substitute $r = 2|E|/3$ in Euler's formula to get $|E| = 3n - 6$, proving (1).

In particular, K_5 has 10 edges, exceeding the maximum number of $3 \times 5 - 6 = 9$. Hence, K_5 is not planar and neither is K_n for all $n > 5$. Similarly, for (3), if every vertex has degree 6, then by Euler's formula, $|E| \geq 6n/2 = 3n$, violating (1). Finally for (4), simply replace the relation $2|E| = 3r$ above by $2|E| = 4r$ to get the desired result. $\bigtriangledown$

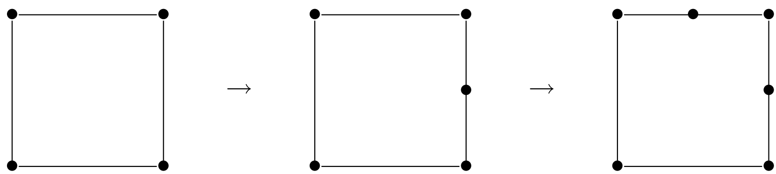
Exercise 5.68. Use Theorem 5.26 to prove that $K_{3,3}$ is not planar.

Question. Can Theorem 5.26 be applied to show that the Peterson graph is not planar?

Exercise* 5.69. Prove that the complement of the Peterson graph is not planar, and neither is the complement of C_n, if $n \geq 8$.

Definition. Let G be a graph. If we replace any edge $ab \in E_G$ by the path $\{av, vb\}$, where v is a new vertex, then the resulting graph is said to be *homeomorphic* to G. More generally, two graphs are *homeomorphic* to each other if one can be obtained from the other by iterating a finite number of replacements in this manner.

For example, we sketch below how to obtain C_6 by applying the procedure twice to C_4. In this way, it is not hard to see that any two cycles are homeomorphic.



Test 5.70. Which one of these four graphs is *not* homeomorphic to any other one?
a) K_3
b) K_4
c) $K_{2,2}$
d) C_9

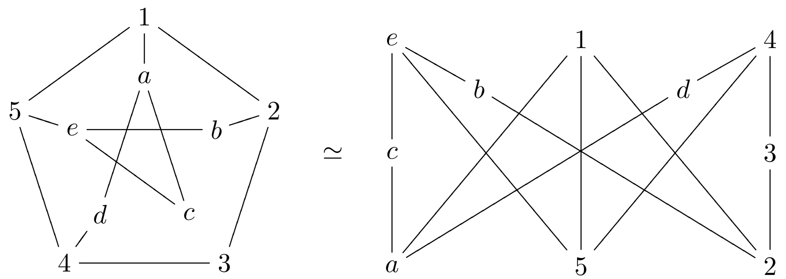
Exercise* 5.71. Show that every graph G is homeomorphic to a bipartite graph, by replacing *every* edge in G by a path of length two.

The next major theorem, due to Kuratowski, reduces the question of planarity to a search for either $K_{3,3}$ or K_5 subgraph. The lengthy proof is omitted, but we shall illustrate how one might apply the theorem to show the non-planarity of the Peterson graph.

Theorem 5.27. A graph is planar if and only if it contains no subgraph homeomorphic to K_5 or $K_{3,3}$.

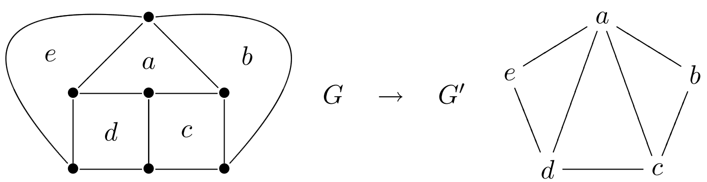
Example. Prove that the Peterson graph is not planar.

Solution. We find a subgraph of the Peterson graph which, after a little modification in the way it is drawn, is shown to be homeomorphic to $K_{3,3}$.



Definition. The *dual graph* G' of a plane graph G is the graph whose vertices are the interior regions of G, and where $rs \in E_{G'}$ if and only if the regions r and s are bounded by a common edge in G.

Below is the illustration of a plane graph G with five interior regions, together with its dual graph G'.



As another example, one may observe that the dual graph of K_4 is K_3.

Test 5.72. Which one of these four is the dual graph of $K_{2,9}$?

a) K_9
b) $K_{8,1}$
c) P_8
d) C_9

Theorem 5.28. The dual graph of any plane graph is planar.

Proof. For each interior region r of the plane graph G, put the vertex r for the dual graph G' somewhere in the middle. To complete the proof, simply observe that we may draw non-intersecting paths from the vertex r to each edge boundary of the region r. $\triangledown$

A plane graph somewhat looks like a world map, in which the interior regions are countries. We say two countries are neighbors when they share a common edge boundary. (Having a common vertex does not make a neighbor.) In fact, map coloring was an early motivation for planar graphs.

Definition. The *chromatic number* of a map is the least number of colors needed to color the countries such that neighbors have distinct colors. In other words, the chromatic number of a map G equals $\chi(G')$.

Exercise* 5.73. Draw the dual graph of the modern day map consisting of the nine countries: Egypt, Iraq, Israel, Jordan, Kuwait, Lebanon, Palestine, Saudi Arabia, and Syria. Then determine the chromatic number.

It was conjectured in the mid nineteenth century that four colors suffice for any map. The four-color theorem remained unproved for over a hundred years, until Appel and Haken gave a complete proof in 1977, involving massive case-by-case figures and extensive computer generated results.

Theorem 5.29 (The Four-Color Theorem)**.** If G is planar then $\chi(G) \leq 4$. Equivalently, the chromatic number of any map is at most four.

Exercise 5.74. Draw a map whose chromatic number equals four.

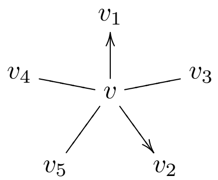
A version of Appel and Haken's proof which includes sufficient details, spans well over 700 pages. We will compromise, in concluding this chapter, with a shorter and nice partial result, first given by Heawood in 1890.

Theorem 5.30. If G is planar then $\chi(G) \leq 5$.

Proof. We define the *color-degree* of a vertex to be the number of distinct colors of vertices which are adjacent to it.

If G has 5 vertices or less, then there is nothing to prove. We proceed by induction, assuming that G has n vertices and that the claim is true for any planar graph with less vertices than n. By Theorem 5.26, G has a vertex v with $\deg(v) \leq 5$. If we ignore v and its associated edges, the resulting subgraph can then be colored with at most 5 colors. Hence, if the color-degree of v is 4 or less, then we have no challenge assigning v to a fifth color to complete the coloring of G with 5 colors or less.

The last case to consider is when $\deg(v) = 5$ with all five adjacent vertices, v_1, v_2, v_3, v_4, v_5, having distinct colors c_i. Without loss of generality, we have labeled these vertices such that v_3 is interior of the region bounded by the rays vv_1 and vv_2, while v_4 lies exterior of it.



Consider the subgraph $G_{1,2}$ of G consisting of all vertices which have been colored either c_1 or c_2, with their associated edges. Note that $v_1, v_2 \in G_{1,2}$. Suppose first that v_1 and v_2 belong to different components in $G_{1,2}$. In the component containing v_1, we swap c_1 and c_2. Doing so does not violate the rules of vertex coloring, but it does decrease the color-degree of v to 4 as v_1 and v_2 now have the same color, i.e., c_2.

But if v_1 and v_2 belong to the same component, then we have a path from v_1 to v_2 and, combined with v_2v, vv_1, a cycle. Since G is planar, this cycle must enclose either v_3 or v_4, but not both. Again, since G is planar, v_3 and v_4 then belong to different components in the subgraph $G_{3,4}$, defined in a similar way. This time, we decrease the color-degree of v by swapping c_3 and c_4 in the component containing v_3, and the proof is complete. $\triangledown$

Books to Read

1. D. L. Applegate, R. E. Bixby, V. Chvátal, and W. J. Cook, *The Traveling Salesman Problem: A Computational Study,* Princeton University Press 2007.

2. G. Chartrand, *Introductory Graph Theory,* Dover Publications 1984.

3. R. Fritsch and G. Fritsch, *The Four-Color Theorem: History, Topological Foundations, and Idea of Proof,* Springer 1998.

4. D. A. Marcus, *Graph Theory: A Problem Oriented Approach,* Mathematical Association of America 2008.