

PHILADELPHIA UNIVERSITY  
DEPARTMENT OF BASIC SCIENCES

Final Exam

Abstract Algebra 2

02–06–2009

Choose any 5 problems from the following 8 problems.

1. Let  $R$  be a commutative ring with unity. (a) What is the meaning of a zero divisor? (b) What is the meaning of a unit element? (c) Prove that a zero divisor is not a unit. (d) Prove that a unit is not a zero divisor.
2. (a) What is the meaning of a ring homomorphism  $\theta : R \rightarrow R'$ ? (b) What is the meaning of  $\ker(\theta)$ ? (c) Prove that  $\ker(\theta)$  is an ideal of  $R$ . (d) Prove that  $\ker(\theta) = \{0\}$  if and only if  $\theta$  is one-to-one.
3. (a) What is the meaning of an ideal? (b) What is the meaning of a principal ideal? (c) Let  $F$  be a field and  $f, g \in F[x]$ . Prove that the set  $I = \{af + bg \mid a, b \in F[x]\}$  is an ideal of  $F[x]$ . (d) Is  $I$  a principal ideal?
4. (a) What is the meaning of an extension field  $K$  over  $F$ ? (b) What is the meaning of an algebraic element  $a \in K$  over  $F$ ? (c) What is the meaning of the minimal polynomial of the element  $a$  over  $F$ ? (d) Find the minimal polynomial of  $a = 1 + \sqrt[3]{2} \in \mathbf{R}$  over  $\mathbf{Q}$ .
5. Let  $p$  be a prime number. Prove that  $\prod_{a \in \mathbf{Z}_p} (x - a) = x^p - x \in \mathbf{Z}_p[x]$ .
6. Prove that  $\mathbf{Q}(1 + i)$  is isomorphic to  $\mathbf{Q}(1 - i)$ , where  $i = \sqrt{-1} \in \mathbf{C}$ .
7. Using  $f = x^3 + x^2 + 1 \in \mathbf{Z}_2[x]$ , draw the addition and multiplication tables for the finite field  $\mathbf{Z}_2[x]/(f)$ .
8. Let  $F$  be a field with 125 elements. Let  $a \in F$  and  $a \notin \mathbf{Z}_5$ . Prove that  $\mathbf{Z}_5(a) = F$ .

**Notes:**

1. Full credit will only be given to a solution which is logically correct. Be very careful in what you write!
2. You may assume all the theorems given in the notes, unless when the problem asks you to prove the theorem.
3. Do not spend too much time on a single problem. Read the entire set of problems first; mark the ones you know how to solve and cross out the ones you don't.
4. Do exactly four problems. No bonus points will be given to a sixth solution and beyond. If you have extra time, double check your work.

–Amin Witno