Philadelphia University
Department of Basic Sciences

Final Exam **Abstract Algebra 2** **08–06–2011**

Choose only 5 problems from the following 8 problems.

1. Let $R = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$.
   (a) Prove that R is a subring of $\mathbb{R}$.
   (b) Prove that R is an integral domain.
   (c) Prove that R is not a field.
2. (a) Find all the units in $\mathbb{Z}$, $\mathbb{Z}_{18}$, and $\mathbb{Q}[x]$.
   (b) Find all the zero divisors in $\mathbb{Z}_3 \times \mathbb{Z}_5$.
   (c) Find one example of a zero divisor in $M(2, \mathbb{Z})$.
3. Let R be a commutative ring, $a \in R$, and $I = \{r \in R \mid ar = 0\}$.
   (a) Prove that I is an ideal of R.
   (b) If $R = \mathbb{Z}_{12}$ and $a = 3$, find the elements of I and R/I.
   (c) Construct the Cayley table for R/I in (b) under multiplication.
4. Let $S = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \middle| \; a, b \in \mathbb{R} \right\}$ and $\mathbb{C}$ be the field of complex numbers.
   (a) Prove that S is a subring of $M(2, \mathbb{R})$.
   (b) Prove that $\mathbb{C} \approx S$ by defining $\theta(a + bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.
5. Prove that $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2} + \sqrt{3})$.
6. Let $f = x^4 - x^2 + 2x - 1 \in \mathbb{Z}_5$.
   (a) Prove that f is reducible in $\mathbb{Z}_5$.
   (b) Factor f using irreducible polynomials.
   (c) Prove that each factor in (b) is irreducible in $\mathbb{Z}_5$.
7. Let $f = x^2 + x + 2 \in \mathbb{Z}_3[x]$ and $F = \mathbb{Z}_3[x]/(f)$.
   (a) Find all the elements of F.
   (b) Prove that F is a field.
   (c) Find the order of $(f) + x + 2$ in the group F^*.
8. Let F be a field with $\chi(F) = 2$. Suppose that F has dimension 5 as a vector space over $\mathbb{Z}_2$. Prove that $F^* = \langle a \rangle$ for any nonzero element $a \neq 1$.

–Amin Witno